ON THE QUADRATURE OF CURVES AND THE CUBATURE OF SOLIDS BY MEANS OF INFINITE SERIES.

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WE shall need the sum of the series

$$1^p+2^p+3^p+\ldots+n^p=A_0n^{p+1}+\ldots \text{ suppose.}$$

To find A_0 — for the coefficients of the remaining terms of the *sum* we shall not need,—let $n=n+1$, then we shall have

$$1^p+2^p+\ldots+n^p+(n+1)^p=A_0(n+1)^{p+1}+\ldots=(n+1)^p+A_0n^{p+1}+$$
$$\therefore n^p+pn^{p-1}+\ldots=A_0[(n+1)^{p+1}-n^{p+1}]+\ldots=(p+1)A_0n^p+\ldots$$

Equating the coefficients of like powers of n, we find

$$A_0=\frac{1}{p+1}. \text{ Hence } 1^p+2^p+\ldots+n^p=\frac{n^{p+1}}{p+1}+\ldots \qquad (1)$$

Wherever the first member of (1) occurs in what follows, it is sufficient to put in its place the the term given in the second member of (1); for we shall always have n^{p+1} multiplied by x'^{p+1}, n being infinitely great and x' infinitely small, and $nx'=$ some finite quantity. Since the remaining terms in the second member of (1) will contain the powers n^p, n^{p-1}, &c., and these will be multiplied by x'^{p+1}, it will be seen that

$$n^p.x'^{p+1}=x'\times \text{finite quantity}=0, \text{ and so of the other terms.}$$

I. THE CIRCLE.

Let the origin be at the centre of the circle, and let ordinates be drawn perpendicular to the radius CD dividing the quadrant ACD into rectangles of infinitely small width, each equal to x'. Let $CB=x=nx'$, and call the area of $ACBP=A$, the radius of the circle being unity. We know from Elementary Geometry that the area of the quadrant ACD equals $\frac{1}{4}\pi$, $\pi=3.14159\ldots$ It will now be seen that

$$A=x'[y_1+y_2+\ldots y_n]=x'[\sqrt{(1-x'^2)}+\sqrt{(1-4x'^2)}+\ldots+\sqrt{(1-n^2x'^2)}..] \quad (2)$$

since $y_n^2=1-n^2x'^2$, n being an integer.

If we expand the radicals in the third member of (2) and arrange the terms, we shall have $(1+1+\ldots=n)$

$$A=x'[n-\tfrac{1}{2}x'^2(1^2+2^2+\ldots+n^2)-\tfrac{1}{2.4}x'^4(1^4+2^4+\ldots+n^4)-\tfrac{3}{2.4.6}x'^6(1^6+2^6+..)..]$$
$$=x-\frac{x^3}{2.3}-\frac{x^5}{2.4.5}-\frac{3x^7}{2.4.6.7}-\frac{3.5\,x^9}{2.4.6.8.9}-\ldots \qquad (3)$$

If we make $x = 1 = CD$, we shall have

$$\tfrac{1}{4}\pi = 1 - \frac{1}{2.3} - \frac{1}{2.4.5} - \frac{3}{2.4.6.7} - \dots \tag{4}$$

Now make the angle $ACP = \theta$, and we find

$A = \tfrac{1}{2}\theta + \tfrac{1}{2}\sin\theta\cos\theta = \tfrac{1}{2}\theta + \tfrac{1}{2}x\sqrt{(1-x^2)}$. Whence, by (3),

$$\theta = 2\left[x - \frac{x^3}{2.3} - \frac{x^5}{2.4.5} - \frac{3x^7}{2.4.6.7} - \dots - \tfrac{1}{2}x\sqrt{(1-x^2)}\right] = 2[(1-\tfrac{1}{2})x - \tfrac{1}{2}(\tfrac{1}{3}-\tfrac{1}{2})x^3 - ..]$$

$$= x + \frac{x^3}{2.3} + \frac{3x^5}{2.4.5} + \frac{3.5x^7}{2.4.6.7} + \dots \tag{5}$$

If this series be reverted we shall have

$$x = \sin\theta = \theta - \frac{\theta^3}{1.2.3} + \frac{\theta^5}{1.2.3.4.5} - \dots \tag{6}$$

If we divide equation (6) by θ, and then make $\theta = 0$, we see that

$$\frac{\sin 0}{0} = 1. \tag{7}$$

If we make $\sin\theta = \theta z$, $z = 1 - \frac{1}{1.2.3}\theta^2 + \dots$, we have $\cos\theta = \sqrt{(1-\theta^2 z^2)}$ $= 1 - \tfrac{1}{2}\theta^2 z^2 - \dots$, $\cos\theta - 1 = -\tfrac{1}{2}\theta^2 z^2 - \dots$, and $(\cos\theta - 1) \div \theta = -\tfrac{1}{2}\theta z^2$.

If $\theta = 0$,
$$\frac{\cos 0 - 1}{0} = \frac{1-1}{0} = \frac{0}{0} = 0. \tag{8}$$

If in (6) we make $\theta = \theta + h$, we shall have

$$\sin(\theta+h) = \sin\theta\cos h + \cos\theta\sin h = \theta + h - \frac{(\theta+h)^3}{1.2.3} + \frac{(\theta+h)^5}{1.2.3.4.5} - \dots$$

If we now subtract (6) from this equation, divide by h and then make $h = 0$, we shall have

$$\sin\theta\left(\frac{\cos 0 - 1}{0}\right) + \cos\theta\frac{\sin 0}{0} = 1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} - \dots;$$

or by (7) and (8),

$$\cos\theta = 1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} - \dots \tag{9}$$

If $t = \tan\theta$, then $\sin\theta = x = t \div \sqrt{(1+t^2)}$. If this value of x be put in (5), and the second member be reduced, we shall find

$$\theta = t - \tfrac{1}{3}t^3 + \tfrac{1}{5}t^5 - \dots \tag{10}$$

The reversion of this will give t in terms of θ.

II. The Ellipse.

The equation of the ellipse is

$$ay = b\sqrt{(a^2-x^2)} = ab\sqrt{(1-x_1^2)},\ x = ax_1. \tag{11}$$

Area $= ax'_1[y_1 + y_2 + \dots y_n] = abx'_1[\sqrt{(1-x_1^2)} + \sqrt{(1-4x_1^2)} + \dots \sqrt{(1-n^2x_1^2)}]$ $= abA$ by (3). When $x = 1$, Eq. (4), $\tfrac{1}{4}$ area of ellipse $= \tfrac{1}{4}\pi ab$. (12)

III. The Parabola.

Let the axis of the parabola be vertical, and also be the axis of y, and let h be the greatest value of y, a being the greatest value of x. The equation of the curve will be

$$x^2 = 2p(h-y), \text{ or } y = h - \frac{x^2}{2p}. \tag{13}$$

Since $nx' = a$, the area of the curve is

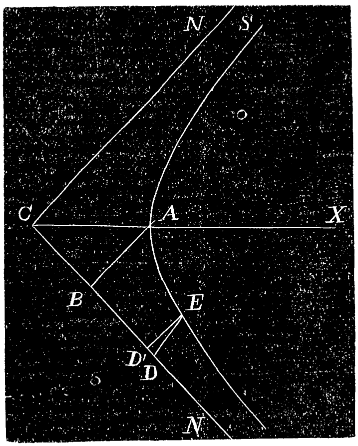
$$A = 2x'[y_1 + y_2 + \ldots y_n] = 2x'[nh - \frac{x'^2}{2p}(1+2^2+\ldots n^2)]$$

$$= 2ah - \frac{a^3}{3p} = \tfrac{4}{3}ah = \tfrac{2}{3} \text{ circumscribing rectangle.} \tag{14}$$

IV. The Hyperbola.

Let SAE be one branch of a hyperbola, CAX the axis, CN the asymptote. If $CD = x$, ED parallel to CN, $ED' = y$, perpendicular to CN; if a and b are the semi-axes, then we have

$$2xy = ab \tag{15}$$



The area of $ABDE = x'(y_1 + y_2 + \ldots y_n)$

$$= \tfrac{1}{2}abx'\left[\frac{1}{x_0+x'} + \ldots + \frac{1}{x_0+nx'}\right]$$

$$= \frac{abx'}{2x_0}\left[\frac{1}{1+\frac{x'}{x_0}} + \frac{1}{1+\frac{2x'}{x_0}} + \ldots + \frac{1}{1+\frac{nx'}{x_0}}\right]$$

$$= \frac{abx'}{2x_0}\left[n - \frac{x'}{x_0}(1+2+\ldots n) + \frac{x'^2}{x_0^2}(1+2^2\ldots)\right]$$

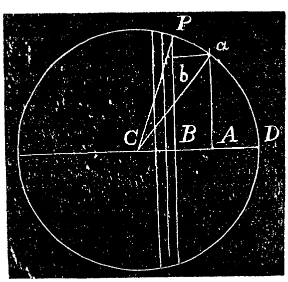
$$= \frac{ab}{2x_0}\left[nx' - \frac{x'^2n^2}{2} + \frac{x'^3n^3}{3} - \ldots\right] = \frac{ab}{2}\left[\frac{x-x_0}{x_0} - \frac{(x-x_0)^2}{2x_0^2} + \frac{(x-x_0)^3}{3x_0^3} - \ldots\right]$$

$$= \frac{ab}{2}\log\left(1 + \frac{x-x_0}{x^0}\right) = \frac{ab}{2}\log\frac{x}{x_0}, \tag{16}$$

where $x_0 = CB$. It is easy now to find the area of the hyperbola.

V. The Sphere.

To find the surface of the sphere conceive it to be divided by planes perpendicular to a diameter into elementary solids (frusta of cones) the thickness, or height of each being x', and let the slant height of these be s_1, s_2, &c. Then the surface of a hemi-sphere will be, y being an ordinate,



$$S = 2\pi[s_1y_1 + s_2y_2 + \ldots + s_ny_n]. \tag{17}$$

Let $ABaP$ be one of the elementary frustums,

$AB = ab = x'$, $aP = s$, $BP = y$, $CP = r =$ radius of the sphere. If ab is perpendicular to Aa the two triangles abP and aAC will be similar and give $sy = rx'$. Hence $s_1y_1 = s_2y_2 = \ldots = s_ny_n = rx'$. This in (17) gives

$$S = 2\pi rnx' = 2\pi rx, \tag{18}$$

where $nx' = x = CA$. If $x = r$, this gives for the surface of the hemisphere $2\pi r^2$, and of the whole sphere, $4\pi r^2$.

VI. The Prolate Spheroid.

To find the surface of the prolate spheroid, conceive it divided, as in the case of the sphere, by planes perpendicular to the transverse axis. Then the surface of a hemi-sphere (or a part of it) will be

$$S = 2\pi[s_1y_1 + s_2y_2 + \ldots + s_ny_n]. \tag{19}$$

If in the last fig. we consider PC a normal, N, we shall have $sy = Nx'$, and

$$S = 2\pi x'[N_1 + N_2 + \ldots + N_n]. \tag{20}$$

But $N = b\sqrt{[1-(e^2x^2 \div a^2)]}$. Hence

$$S = 2\pi bx'\left[\sqrt{\left(1-\frac{e^2x'^2}{a^2}\right)}+\sqrt{\left(1+\frac{4e^2x'^2}{a^2}\right)}+\ldots+\sqrt{\left(1-\frac{n^2e^2x'^2}{a^2}\right)}\right]. \tag{21}$$

Suppose in the first figure that $CB = ex \div a = nex' \div a = \sin\theta$; then, as in Prob. II,

$$\frac{ex'}{a}\left[\sqrt{\left(1-\frac{e^2x'^2}{a^2}\right)}+\ldots+\sqrt{\left(1-\frac{n^2e^2x'^2}{a^2}\right)}\right. = \tfrac{1}{2}\theta + \tfrac{1}{2}\sin\theta\cos\theta$$

$$= \tfrac{1}{2}\sin^{-1}\frac{ex}{a} + \frac{ex}{2a}\sqrt{\left(1-\frac{e^2x^2}{a^2}\right)}.$$

We therefore have for that portion of the surface of the prolate spheroid, which is included between two planes drawn perpendicular to the transverse axis of the generating ellipse, one through the centre and the other at the distance x from it,

$$S = \frac{\pi ab}{e}\left[\sin^{-1}\frac{ex}{a} + \frac{ex}{a}\sqrt{\left(1-\frac{e^2x^2}{a^2}\right)}\right]. \tag{22}$$

If $x = a$, surface of whole spheroid equals

$$2S' = 2\pi ab\left[\frac{1}{e}\sin^{-1}e + \sqrt{(1-e^2)}\right]. \tag{23}$$

VII. The Oblate Spheroid.

Equation (20) will give the surface of the oblate spheroid if we make $N = a\sqrt{[1+(a^2e^2x^2 \div b^4)]}$, and $b^2z = aex$, so that

$$S = \frac{2\pi b^2z'}{ae}\left[\sqrt{(1+z'^2)}+\sqrt{(1+4z'^2)}+\ldots+\sqrt{(1+n^2z'^2)}\right]. \tag{24}$$

If the terms of (24) be expanded and the result treated as in Prob. II, we shall find, since $nz' = z$,

$$S = \frac{2\pi b^2}{ae}\left[z + \frac{z^3}{2.3} - \frac{z^5}{2.4.5} + \frac{z^7}{2.4.6.7} - \ldots\right]. \tag{25}$$

VIII. The Logarithmic Curve.

If we have $$y = a^x, \tag{26}$$
area of curve $= A = x'[y_1+y_2+\ldots+y_n] = x'[a^{x'}+a^{2x'}+\ldots+a^{nx'}]$
$$= x'\left[n+\lambda x'(1+2+\ldots+n)+\frac{\lambda^2 x'^2}{1.2}(1+4+\ldots+n^2)+\ldots\right]$$
$$= x+\frac{\lambda x^2}{1.2}+\frac{\lambda^2 x^3}{1.2.3}+\ldots = \frac{1}{\lambda}\left(a^x-1\right). \quad (\lambda = \log a.) \tag{27}$$

IX. The Ellipsoid.

To find the solid contents of the ellipsoid, suppose sections made by planes perpendicular to the axis of z, the distance between the planes being z'. Each section will be an ellipse whose axes we shall call $2x$ and $2y$. The area of a section will then be πxy. The solid contents of a portion of the ellipsoid will now be
$$S = \pi z'[x_1 y_1+x_2 y_2+\ldots+x_n y_n]. \tag{28}$$
Now let two sections be made, one passing through the axes of x and z, and the other through y and z, each section will be an ellipse, giving
$$cx = a\sqrt{(c^2-z^2)}, \qquad cy = b\sqrt{(c^2-z^2)} \tag{29}$$
The product of these will be
$$c^2 xy = ab(c^2-z^2). \tag{30}$$
We now have, since $nz' = z$,
$$S = \frac{\pi abz'}{c^2}\left[nc^2 - z'^2(1+4+\ldots n^2)\right] = \frac{\pi ab}{c^2}\left(c^2 z-\frac{z^3}{3}\right). \tag{31}$$
If $z = c$, $2S = \frac{4}{3}\pi abc$, (32). This solution includes the sphere and the spheroids.

It is easy to extend this method to the paraboloid, hyperboloid of revolution, elliptical paraboloid, and a great variety of other cases, but the process is so easy that we need not work out these cases here. We see here how the Integral Calculus sorts out the terms of series which are needed in the final result.

The surface of the hyperboloid of revolution can be found without difficulty. For this case, N in Eq. (20) is equal to $b\sqrt{[(e^2x^2 \div a^2)-1]} = b\sqrt{(z^2-1)}$, $az = ex$. Hence
$$S = \frac{2\pi abz'}{e}\left[\sqrt{(z'^2-1)}+\sqrt{(4z'^2-1)}+\ldots+\sqrt{(n^2z'^2-1)}\right].$$
By using Prob. IV, we can find this equal to
$$S = \pi b\left[\frac{x}{a}\sqrt{(e^2x^2-a^2)}-b+\frac{a}{e}\log\left(\frac{ex+\sqrt{(e^2x^2-a^2)}}{ae+b}\right)\right].$$
We can also easily find the surface of the paraboloid of revolution.